Generalized symmetric supermodular functions

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Abstract

This work singles out and examines the downgrading property of supermodular functions which is crucial to several results in lattice programming. We introduce and study classes of functions which satisfy some variant of the downgrading property and therefore preserve most of the properties of supermodular functions which are of interest to lattice programming.

1 Introduction

Initiated by Topkis and Veinott about twenty years ago, lattice programming has by now greatly increased its importance as a technique to obtain qualitative insights in several different areas of research, ranging from Economics to Operations Research. Despite some early generalizations (see [10]), however, most of its applications until recently have been confined to just a few classes of functions characterized by some mathematical property.

The main instance of such property has an ubiquitous nature and it has been used under different names in several fields. In the area of lattice programming, the prevalent terminology refers to it as supermodularity or superadditivity. The first term was proposed by Edmonds and is related to some questions in combinatorial optimization. The second one has been especially advocated by Veinott. Unfortunately, the first choice bears no linguistic relationships with its rôle in lattice programming, while the second one conflicts with different usages of the same term. While regretting the absence of a satisfactory established terminology , we will choose to use the term "supermodularity" whose link to lattice programming seems better entrenched.

Ongoing work from Veinott [11] and Milgrom, Roberts and Shannon [6] has made increasingly clear that most results of lattice programming can be generalized to a class of functions far larger than the supermodular ones. Unsurprisingly, such class of generalized supermodular functions departs even further from the generalizations currently under study in combinatorial optimization (see for instance [7]), indicating a possible fading

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of the initial overlapping between lattice programming and combinatorial optimization where supermodular functions originated (see [4]).

The purpose of this work is to single out a class of generalized supermodular functions whose properties are fundamental for lattice programming and to study some subclasses of these. To this aim, we derive a property of supermodular functions (independently noticed also by Milgrom, Roberts and Shannon [6]), which appears to be crucial for some key results in lattice programming. We call this the downgrading property, since it can be interpreted as a condition guaranteeing that if a function decreases when moving in some particular direction, it must also decrease when moving in another corresponding direction. The kind of decrease specifies five different variants of the downgrading property. We then define a few subclasses of functions satisfying some version of the downgrading property. In particular, we examine the relationships among subclasses satisfying a symmetry condition.

The work is organized as follows. The next section introduces several preliminary definitions and some notation. Section 2 examines the downgrading property and considers its rôle in four key results of lattice programming. Section 3 studies some subclasses of generalized supermodular functions, with particular reference to the symmetric ones.

1.1 Preliminary definitions and notation

A *partial order relation* on a set X is a binary relation $\succeq: X \times X \to X$ which is reflexive, transitive and antisymmetric. A *partially ordered set* $(X; \succeq)$ consists of a nonempty set X endowed with a partial order relation $\succeq$. When there is not ambiguity, we will say for short that X (rather than $(X; \succeq)$) is a *poset*, meaning that the partial order relation is understood. In particular, unless differently specified, $\mathbf{R}$ will always be assumed to be endowed with the usual $\geq$ order relation. The derived binary relation $\succ$, $\preceq$ and $\prec$ are defined in the obvious way. If X is a poset, we say that the elements of the pair $(x, y) \in X \times X$ are *comparable* if $x \succeq y$ or $y \succeq x$ (or both). If x, y are not comparable, we write $x \| y$.

Let S be a subset of a poset X. We call S a *subposet*. An element $x \in X$ is said to be an *upper bound* for S if $x \succeq y, \forall y \in S$. An upper bound x of S is called the *supremum* of S if, for any upper bound y of S, it is $y \succeq x$. It is easy to prove that, if it exists, the supremum of S is unique; we will denote it by $x = \vee S$. *Lower bounds* and *infima* are analogously defined; the infimum of a set S is written $\wedge S$. Such similarity in definitions stems from a more general *duality principle*, stating that if we define $\preceq$ as the dual ordering of $\succeq$, any statement about a poset $(X; \succeq)$ is true in its dual poset $(X; \preceq)$.

If it exists, we denote by their *join* $x \vee y$ (respectively *meet* $x \wedge y$) the supremum (infimum) of two elements x, y of a poset X. If a subset L of a poset X contains at least the meet or the join of each pair of its elements, we call it a *subquasilattice*. If it always contains the join (dually, the meet) of each pair of its elements, we say that L is a *join sublattice* (*meet sublattice*). If L is either a join sublattice or a meet sublattice, we call it a *subsemilattice*. If it contains both the join and the meet of each pair of its elements, we call it a *sublattice*. Finally, if all its elements are comparable, we say that L is a *subchain*.

If a poset X is a sublattice of itself, we omit the prefix "sub" and simply call it a *lattice*. Quasilattices, join and meet lattices, semilattices, and chains are analogously defined. A (sub)chain is a (sub)lattice; a (sub)lattice is a join (sub)lattice and a meet (sub)lattice; any of these latter two is a (sub)semilattice; and a (sub)semilattice is a (sub)quasilattice.

A (sub)poset X is *chain complete* if $\vee S$ and $\wedge S$ exist and are contained in X for any nonempty subchain $S \subseteq X$. A (sub)lattice L is *complete* if $\vee S$ and $\wedge S$ exist and are contained in L for any nonempty subset $S \subseteq L$. A (sub)lattice is complete if and only if it is chain complete. Given a lattice L and a function $f : L \to \mathbf{R}$, we define for any $\alpha \in \mathbf{R}$ its *upper set of level* α to be $U_\alpha = \{x \in L : f(x) \geq \alpha\}$ and we say that f is *upper chain complete* if all its upper level sets (which are not necessarily sublattices) are chain complete. Given a (chain) complete lattice L, we say that the function $f : L \to \mathbf{R}$ is *order upper semicontinuous* if $\limsup_{x\in C, x\downarrow\wedge C} f(x) \leq f(\wedge C)$ and $\limsup_{x\in C, x\uparrow\vee C} f(x) \leq f(\vee C)$, for any chain C in L.

Given two comparable elements $y \succeq x$ in a lattice L, we define the *open interval* (x, y) as the set $\{z \in L : y \succ z \succ x\}$ and the *closed interval* $[x, y]$ as the set $\{z \in L : y \succeq z \succeq x\}$. We speak generally of an *interval* when it is not specified whether the endpoints belong to the interval. Given a finite number of posets $(X_i, \succeq_i)$ $(i = 1, \ldots, n)$ we define their *direct product* to be the set $L = \{x = (x_1, \ldots, x_n) : x_i \in X_i; i = 1, \ldots, n\}$ ordered by the rule that $x \succeq y$ if and only $x_i \succeq_i y_i$, for all $i = 1, \ldots, n$. The direct product of n lattices is still a lattice. In Section 3.5, we will consider the direct product of n chains. The most common example of this is the lattice $\mathbf{R}^n$ with the natural componentwise ordering $\geq$ given by $x \geq y$ if and only if $x_i \geq y_i$, for all $i = 1, \ldots, n$. Remark that an interval in $(\mathbf{R}^n; \geq)$ is the direct product of n subchains of $\mathbf{R}$.

Let $f : X \to Y$ be a mapping which associates to any element x of a chain $(X; \succeq_1)$ an element $f(x)$ in another chain $(Y; \succeq_2)$. We say that f is *nondecreasing* if $x \succeq_1 y$ implies $f(x) \succeq_2 f(y)$, for all $x, y \in X$. Similarly, we say that f is *increasing* if $x \succ_1$ implies $f(x) \succ_2 f(y)$, for all $x, y \in X$. Nonincreasing and decreasing functions are analogously defined.

Let $J = <a, b>$ denote any interval in $\mathbf{R}$, where we have used the angular brackets to denote that J can be open or closed on either side. Given a function $f : X \to Y$, let $Ran(f) \subseteq Y$ denote the range of the function f. If $Y = \mathbf{R}$, we denote by $Co(f)$ the convex hull of $Ran(f)$. We denote the extended real line $\mathbf{R} \cup \{-\infty, +\infty\}$ by $\overline{\mathbf{R}}$; analogously, we will write $\overline{\mathbf{R}}^+$ for the extended nonnegative real line $\mathbf{R}^+ \cup \{+\infty\}$ and $\overline{\mathbf{R}}^-$ for the extended nonpositive real line. The algebraic and ordinal properties of $\overline{\mathbf{R}}$ are defined in the usual way.

Let X be a set. A *binary operation* on X is a function $*$ from $X \times X$ into X. If the restriction to $X \times X$ of a function $*$ is a binary operation on X, then we say that $*$ is *closed under* $*$. If there exists an element $e \in X$ such that $x * e = e * x = x$ for all $x \in X$, we say that e is an *identity* for $*$. If there exists an element $o \in X$ such that $x * o = o * x = o$ for all $x \in X$, we say that o is a *null element* for $*$. If they exist, both identity and null element are unique.

If $*$ is a binary operation on X, we define for each element $x \in X$ the *horizontal section*

$h^*_x : X \to X$ by $h^*_x(y) = y{*}x$ and the *vertical section* $v^*_x : X \to X$ by $v^*_x(y) = x{*}y$. If X is also a chain, we say that $*$ is *nondecreasing* if all its sections are nondecreasing and that it is *increasing* if they are increasing. If $*$ has a null element o, it cannot be increasing over the sections h^*_o and v^*_o. Therefore, we call a binary operation *weakly increasing* if all its sections with the possible exception of these two are increasing. An increasing binary operation is weakly increasing; and a weakly increasing one is nondecreasing.

Given the binary operation $*$ on the set X, we say that $*$ is *associative* if $(s*t)*u = s*(t*u)$ for all $s,t,u \in X$; that it is *commutative* if $s*t = t*s$ for all $s,t \in X$; and that it is *medial* if $(s*t)*(u*v) = (s*u)*(t*v)$ for all $s,t,u,v \in X$. This latter property is also known as bisymmetry. Furthermore, we say that $*$ is *idempotent* if $s*s = s$, for all $s \in X$ and that it is *cancellative* if all its horizontal and vertical sections are injective. Finally, if X is the closed interval $[a,b] \subset \mathbf{R}$, we say that $*$ is *triangular* if it is associative, commutative, nondecreasing and has either a or b as identities. It can be shown that if b (respectively, a) is the identity of a triangular operation, then a (b) must be a null element for it. To distinguish these two cases, we speak respectively of *triangular norms* and *conorms.*

2 The downgrading property

2.1 Downgrading functions

We recall the following definition of a class of functions of great interest in lattice programming and combinatorics.

Definition 1 *Let L be a lattice. We say that the function $f : L \to \mathbf{R}$ is supermodular if*

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y) \tag{1}$$

for all $x, y \in L$. If the inequality holds strictly for all $x \| y \in L$, we say that f is strictly supermodular. Similarly, f is said to be (strictly) submodular if $-f$ is (strictly) supermodular.

An equivalent definition of supermodular functions can be given using the local nature of (1). Given a lattice L, a function $f : L \to \mathbf{R}$ is said to be *supermodular at x, y* if $f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$. Then f is supermodular if it is supermodular at x, y for any $x \| y \in L$. Such observation applies to all the functions that we define below. However, for simplicity, we will introduce them always through the global characterization. On the other hand, when necessary, we will say that a function has a given property at x, y if it satisfies that property locally at $x \| y$.

Rearranging (1), one can easily deduce that a supermodular function has the ordinal implication that for all $x, y \in L$, $f(x) \geq f(x \vee y)$ implies $f(x \wedge y) \geq f(y)$ and $f(x) > f(x \vee y)$ implies $f(x \wedge y) > f(y)$. Since it suggests that a nonincreasing order is maintaned across the images of given pairs of values, we will call this the downgrading property. Its dual, corresponding to the case of submodular functions, is called upgrading property.

In several applications of lattice programming, the full power of the assumption of supermodularity is not necessary and it suffices to make use of some variants of the downgrading property, which can be generated as follows. Let R, S denote either $\geq$ or $>$. Then, the general form of the downgrading property states that for all $x \| y \in L$, $f(x) R f(x \vee y)$ implies $f(x \wedge y) S f(y)$. It is not difficult to see that there are only five possible combinations which can be used to define some form of downgrading property. This motivates the introduction of the following classes of functions.

Definition 2 *Let L be a lattice. We say that the function $f : L \to \mathbf{R}$ is: 1) downgrading if $f(x) \geq f(x \vee y)$ implies $f(x \wedge y) \geq f(y)$ and $f(x) > f(x \vee y)$ implies $f(x \wedge y) > f(y)$ for all $x, y \in L$; 2) strictly downgrading if $f(x) \geq f(x \vee y)$ implies $f(x \wedge y) > f(y)$ for all $x \| y \in L$; 3) meet-downgrading if $f(x) \geq f(x \vee y)$ implies $f(x \wedge y) \geq f(y)$ for all $x, y \in L$; 4) join-downgrading if $f(x) > f(x \vee y)$ implies $f(x \wedge y) > f(y)$ for all $x, y \in L$; and 5) quasidowngrading if $f(x) > f(x \vee y)$ implies $f(x \wedge y) \geq f(y)$ for all $x, y \in L$.*

It is not difficult to check that a strictly downgrading function is also downgrading; that a function is downgrading if and only if it is both meet and join-downgrading; and that meet and join-downgrading functions are quasidowngrading. An important alternative characterization of the quasidowngrading functions is that their upper level sets are quasisublattices.

2.2 Some consequences of the downgrading property

In order to appreciate the rôle of the downgrading property in lattice programming, we briefly consider the generalizations of four of its key results under various forms of it. Since our goal is simply to show the usefulness of the downgrading property, we will not necessarily state them in their utmost generality. The first result is an existence theorem proved by Veinott in 1976 [12] which shows that under a mild assumption a quasidowngrading function attains its maximum.

Theorem 1 *Let L be a lattice. If $f : L \to \mathbf{R}$ is quasidowngrading and upper chain complete, then f attains its maximum on L.*

We remark that if L is complete and f is order upper semicontinuous, then f is upper chain complete. This provides a useful sufficient condition to guarantee that Theorem 1 holds. The second result we present characterizes the set of maximizers of a downgrading function and extends a theorem of Topkis [9].

Theorem 2 *Let L be a lattice and $f : L \to \mathbf{R}$. Then the following is true for the set M of its maximizers: 1) if f is strictly downgrading, M is a subchain; 2) if f is downgrading, M is a sublattice; 3) if f is join-downgrading (respectively, meet-downgrading), M is a join sublattice (meet sublattice); 4) if f is quasidowngrading, M is a quasisublattice.*

Proof: Since all the proofs are analogous, we only prove the case of a downgrading function. Assume $x \| y$ are maximizers. Then, $f(x) = f(y)$. If $x \vee y$ is not a maximizer, then $f(x) > f(x \vee y)$ and the downgrading property implies $f(x \wedge y) > f(y)$, which contradicts the assumption that y is a maximizer. Hence, $x \vee y$ must be a maximizer. A similar argument shows that $x \wedge y$ is also a maximizer. $\square$

Remark that the fourth statement follows immediately from the observation that a function is quasidowngrading if and only if all its upper level sets are subquasilattices. The next result concers preservation of the downgrading property under maximization and generalizes a theorem given in [2]. To present it, we introduce some notation. Let X, A be posets and $A_x \subseteq A$ be nonempty for all $x \in X$. Order (partially) the set $L = \{(x, a) : x \in X,\, a \in A_x\}$ with the ordering induced from the direct product of X and A. Given a function $f : L \to \mathbf{R}$, define the maximum function $F : X \to \mathbf{R}$ by $F(x) = \sup_{a \in A_x} f(x, a)$.

Theorem 3 *Let L be a lattice and $F : X \to \mathbf{R}$ finite on X. If $f : L \to \mathbf{R}$ has a variant of the downgrading property, this same variant is inherited by F.*

Proof: We only prove the case of a downgrading function. Since L is a lattice, it follows that both X and A_x (for all $x \in X$) are also lattices. Choose arbitrary x_1, x_2 in X. If x_1 and x_2 are comparable, the argument is trivial; so, assume $x_1 \| x_2$. By definition, for any $\epsilon > 0$, there exist $a_i \in A_{x_i}$ such that $F(x_i) < f(x_i, a_i) + \epsilon$, for $i = 1, 2$.

Since L is a lattice, both $(x_1, a_1) \vee (x_2, a_2) = (x_1 \vee x_2, a_1 \vee a_2)$ and $(x_1, a_1) \wedge (x_2, a_2) = (x_1 \wedge x_2, a_1 \wedge a_2)$ are in L. Therefore, we have

$$f(x_1, a_1) - f(x_1 \vee x_2, a_1 \vee a_2) \geq F(x_1) - F(x_1 \vee x_2) - \epsilon \tag{2}$$

$$F(x_1 \wedge x_2) - F(x_2) \geq f(x_1 \wedge x_2, a_1 \wedge a_2) - f(x_2, a_2) - \epsilon \tag{3}$$

and

$$f(x_2, a_2) - f(x_1 \wedge x_2, a_1 \wedge a_2) \geq F(x_2) - F(x_1 \wedge x_2) - \epsilon \tag{4}$$

$$F(x_1 \vee x_2) - F(x_1) \geq f(x_1 \vee x_2, a_1 \vee a_2) - f(x_1, a_1) - \epsilon \tag{5}$$

Assume $F(x_1) \geq F(x_1 \vee x_2)$. Let $\epsilon \to 0$, and using the downgrading property of f conclude from (2) and (3) that $F(x_1 \wedge x_2) \geq F(x_2)$. Similarly, one obtains from (4) and (5) that if $F(x_1) > F(x_1 \vee x_2)$, then $F(x_1 \wedge x_2) > F(x_2)$. $\square$

Notice that by Theorem 1 it suffices to assume that f is upper chain complete on L to guarantee that F is finite on X. The fourth result is a monotonicity theorem which particularizes a result of Veinott [11] about increasing selections from ascending multipofunctions. Given a poset X and a lattice A, denote by $L(A)$ the set of all sublattices of A. We say that a map $\Gamma : X \to L(A)$ is *ascending* on X if for all $x_1 \succeq x_2$ in X it holds that $x_1 \vee x_2 \in \Gamma(x_1)$ and $x_1 \wedge x_2 \in \Gamma(x_2)$. Furthermore, if $y_1 \succeq y_2$ for all $y_i \in M(x_i)$ $(i = 1, 2)$, we say that $\Gamma(x)$ is *strictly ascending* on X. Under the same conventions preceding Theorem 3, further assume that A_x is a lattice for all $x \in X$. For each $x \in X$, denote the (parameterized) set of maximizers for $f(x)$ by $M(x) = \{a \in A_x : f(x, a) = F(x)\}$. Define $X' = \{x \in X : M(x) \text{ is nonempty}\}$.

Theorem 4 *Let L be a lattice. If A_x is ascending on X and $f : L \to \mathbf{R}$ is (strictly) downgrading, then the set of maximizers $M(x)$ for f is (strictly) ascending on X'.*

Proof: Choose arbitrary $x_1 \succeq x_2$ in X' and corresponding $a_i \in M(x_i)$ $(i = 1, 2)$. Since A_x is ascending on X, $a_1 \vee a_2 \in A_{x_1}$ and $a_1 \wedge a_2 \in a_{x_2}$. Therefore, $(x_1, a_1 \vee a_2) \in L$ and $(x_1, a_1 \wedge a_2) \in L$. Since $a_1 \in M(x_1)$, $f(x_1, a_1) \geq f(x_1, a_1 \vee a_2)$. Thus, by the downgrading property, $f(x_2, a_1 \wedge a_2) \geq f(x_2, a_2) = F(x_2)$ and therefore $a_1 \wedge a_2 \in M(x_2)$. Similarly, since $a_2 \in M(x_2)$, $f(x_2, a_2) \geq f(x_2, a_1 \wedge a_2)$. Therefore $f(x_1, a_1 \wedge a_2) \geq f(x_1, a_1) = F(x_1)$ and $a_1 \vee a_2 \in M(x_1)$.

Assume further that f is strictly downgrading. Then a_1 and a_2 must be comparable, otherwise $f(x_1, a_1) \geq f(x_1, a_1 \vee a_2)$ would imply $f(x_2, a_1 \wedge a_2) > f(x_2, a_2) = F(x_2)$ which is impossible. $\square$

By Theorem 1, if we assume that f is upper chain complete we can strenghten the conclusion to hold on X itself. Also, we remark that weaker forms of this theorem can be given for weaker variants of the downgrading property.

3 Generalized supermodular functions

3.1 Super$*$ functions

The importance of the downgrading property justifies the interest to define classes of functions satisfying it which are larger than the class of supermodular functions. This motivates the following definition.

Definition 3 *Let L be a lattice. We say that the function $f : L \to Ran(f)$ is super$*$ if there exists a (nonconstant) weakly increasing binary operation $*$ on $Co(f)$ such that*

$$f(x \vee y) * f(x \wedge y) \geq f(x) * f(y) \tag{6}$$

for all $x, y \in L$. If the inequality holds strictly for all $x \| y \in L$, we say that f is strictly super$*$.

The most frequent examples of super$*$ functions in the literature are the supermodular (or superadditive) functions for $* = +$ and the *supermultiplicative* functions for $* = \cdot$. Notice however that the $*$ operation is not supposed to be necessarily commutative or continuous.

Veinott [10] has introduced super$*$ functions under the weaker requirement that $*$ is nondecreasing. The main advantage of such choice is that it allows one to include in the class of (strictly) super$*$ functions the (strictly) *supermaximal* and *superminimal* functions, respectively defined by letting $* = \vee$ and $* = \wedge$ in Definition 3 (see Section 3.4 of this work). On the other hand, such an extensive definition makes the class of super$*$ functions too large to be interesting for the purpose of ensuring some version of the downgrading property. In fact, as the following simple example shows, if $*$ is only nondecreasing then a super$*$ function may not even be quasidowngrading.

Let $D = \{0,1,2\}$ and $C = \{0,1,2,3,4,5\}$. Define $f : D \times \{0,1\} \to C$ by $f(0,0) = 0$; $f(1,0) = 1$; $f(2,0) = 4$; $f(0,1) = 3$; $f(1,1) = 2$; $f(1,2) = 5$. Define the nondecreasing binary operation $*$ on $Ran(f)$ by

$$s * t = \begin{cases} s \vee t & \text{if } s + t \geq 5 \\ 1 & \text{otherwise} \end{cases}$$

Then f is super$*$ (for $*$ nondecreasing), but it is not quasidowngrading: for $x = (0,1)$ and $y = (1,0)$, we have $f(x) > f(x \vee y)$ but $f(x \wedge y) \not\geq f(y)$.

For this reason, we choose to allow a slightly more restricted class of functions in Definition 3 at the expense of leaving out supermaximal and superminimal functions, which will be discussed separately in Section 3.4. The most immediate advantage of such choice is that it ensures indeed that some form of the downgrading property always holds for super$*$ functions. The strongest possible result is the following.

Theorem 5 *Let L be a lattice. If $f : L \to \mathbf{R}$ is a super$*$ function, then f is quasidowngrading. Moreover, if f is strictly super$*$, then it is strictly downgrading.*

Proof: We prove the counterpositive. Assume f is not quasidowngrading. Then, there exist x, y such that $f(x) > f(x \vee y)$ and $f(y) > f(x \wedge y)$. By weak increasingness of $*$, it follows that $f(x \vee y) * f(x \wedge y) \leq f(x) * f(y)$ with equality holding only if the null element appears on both sides of the inequality. However, by uniqueness of the null element, this cannot happen and therefore the inequality is strict. Hence, f is not super$*$.

The proof for the strict case is analogous. Assume f is not strictly downgrading. Then there exist $x \| y$ such that $f(x) \geq f(x \vee y)$ and $f(y) \geq f(x \wedge y)$. By weak increasingness of $*$, we have $f(x \vee y) * f(x \wedge y) \leq f(x) * f(y)$ and f is not a strictly super$*$ function. $\square$

We now show that this is indeed the strongest possible result. Consider the following example. Let $f : \{0,1\}^2 \to \{0,1,2\}$ be defined by $f(0,0) = 1$; $f(1,0) = 2$; $f(0,1) = 0$; $f(1,1) = 0$. Then f is supermultiplicative but not meet-downgrading. The same example with $f(0,0) = 0$ and $f(1,1) = 1$ works to show that a supermultiplicative function is not join-downgrading. The problem here is that 0 is a null element for $\cdot$ and therefore the supermultiplicativity property fails to separate elements whose image is 0.

3.2 Symmetric super$*$ functions

Despite the reassuring nature of Theorem 5, in its full generality the problem of characterizing classes of functions which preserve the downgrading property is very complex. As an example, it suffices to notice that any real-valued function f on a lattice L such that there exist two real-valued increasing functions g_1, g_2 on $Ran(f)$ such that

$$g_1 \circ f(x \vee y) + g_2 \circ f(x \wedge y) \geq g_1 \circ f(x) + g_2 \circ f(y) \tag{7}$$

is downgrading. The proof of this fact is similar to the proof of Theorem 9 and it is omitted.

The necessity to simplify the characterization problem and the desire to ensure stronger forms of the downgrading property for super$*$ functions motivate the imposition of additional restrictions on super$*$ functions. Throughout the rest of this paper, we will therefore consider mainly the following class of functions.

Definition 4 *Let L be a lattice. We say that a (strictly) super$*$ function $f : L \to Ran(f)$ is symmetric if the (nonconstant) weakly increasing binary operation $*$ is commutative and continuous on $Co(f)$.*

As the name suggests, the definition of symmetric super$*$ function mainly relies on the commutativity property. The continuity assumption, in fact, is technical and could be relaxed to a certain extent which however we do not believe advantageous to pursue in this work.

We now proceed to consider two main subclasses of symmetric super$*$ functions which are downgrading and for which some characterization results can be given. For the sake of the brevity, we will omit in the sequel to specify that we are only considering symmetric super$*$ functions, unless this is necessary to avoid ambiguities.

Definition 5 *Let L be a lattice. We say that a (strictly) symmetric super$*$ function $f : L \to \mathbf{R}$ is (strictly) superassociative if $*$ is associative on $Co(f)$.*

We notice that although the choice of the name superassociative is meant to stress that the associativity assumption is the most crucial to our development, all the other properties buried in the definition of a symmetric super$*$ function are also important. The same remark applies to the following definition.

Definition 6 *Let L be a lattice. We say that a (strictly) symmetric super$*$ function $f : L \to \mathbf{R}$ is (strictly) supermedial if $*$ is medial on $Co(f)$.*

We consider now a few well-known results from the theory of functional equations. Their proofs can be found for instance in [1] and in [3].

Theorem 6 *Let $J = < a, b >$ be a (possibly unbounded) proper interval in $\mathbf{R}$ and $* : J^2 \to J$ a cancellative nondecreasing binary operation. Then $*$ is continuous and associative if and only if there exists a continuous and increasing function $g : J \to \mathbf{R}$ such that $s * t = g^{-1}[g(s) + g(t)]$. Moreover, this can happen only if J is open at least on one side and $*$ is increasing.*

The last statement follows from the following observations. It can be shown that under the assumptions of Theorem 6 there exists an element s such that it is either $s > s * s$ or $s * s > s$ (of course, there may be two distinct elements such that both conditions hold). If there is a s such that $s * s > s$ the interval must be open at least on the right. If there is a t such that $t > t * t$ the interval must be open at least on the left. Moreover, it can also be shown that the assumptions of cancellativity, nondecreasingness and continuity together imply that $*$ must be increasing. Finally, notice that in this case the property of commutativity is not assumed *a priori*, but it is a consequence of the representation.

Theorem 7 *Let $J = [a, b]$ be a compact interval in* $\mathbf{R}$ *and* $* : J^2 \to J$ *a triangular binary operation. Then* $*$ *is continuous and weakly increasing if and only if there exists a continuous and increasing function* $g : [a, b] \to \overline{\mathbf{R}}$ *such that* $s * t = g^{-1}[g(s) + g(t)]$ *and one of the following holds: 1)* $g(a) = 0$ *and* $g(b) = +\infty$ *(so that g has range* $\overline{\mathbf{R}}^+$*); 2)* $g(a) = -\infty$ *and* $g(b) = 0$ *(so that g has range* $\overline{\mathbf{R}}^-$ *).*

Notice that Theorems 6 and 7 apply respectively when the domain of $*$ is not closed or it is compact. For medial operations, the following characterization theorem holds.

Theorem 8 *Let J be a (possibly unbounded) interval in* $\mathbf{R}$ *and* $* : J^2 \to J$ *a cancellative weakly increasing binary operation. Then* $*$ *is idempotent, continuous and medial if and only if there exists a continuous and increasing function* $g : J \to \mathbf{R}$ *such that*

$$s * t = g^{-1}\left[\frac{g(s) + g(t)}{2}\right]$$

3.3 Supermodularizable functions

We now define a class of symmetric super$*$ functions which satisfies the downgrading property and are amenable to a more complete study of their properties. Their definition can be thought as the symmetric version of Equation (7).

Definition 7 *Let L be a lattice. We say that the function* $f : L \to \mathbf{R}$ *is supermodularizable if there exists an increasing continuos function* $g : Co(f) \to \mathbf{R}$ *such that*

$$g \circ f(x \vee y) + g \circ f(x \wedge y) \geq g \circ f(x) + g \circ f(y) \tag{8}$$

for all $x, y \in L$. If the inequality holds strictly for all $x \| y \in L$, we say that f is strictly supermodularizable.

Remark that this definition is quite reminiscent of that one of concavifiable functions (see [13]). Contrary to the case of these, however, the assumption that g be a continuous function is not necessary. Indeed, most of our results can be derived without making use of it. However, its introduction greatly simplifies the exposition without substantial loss for any insight the following may offer.

Notice that we can always define a binary operation $*$ on $Co(f)$ by letting $s * t = K[g(s) + g(t)]$, where $K \in \mathbf{R}$ is a normalizing constant which is necessary to guarantee that $*$ maps into $Co(f)$. Such $*$ is obviously commutative and therefore any (strictly) supermodularizable function is (strictly) symmetric super$*$. Hence, Theorem 5 guarantees that it is quasidowngrading. However, we can strengthen such result and prove that a supermodularizable function is also downgrading.

Theorem 9 *Let L be a lattice and* $f : L \to \mathbf{R}$ *a (strictly) supermodularizable function. Then f is (strictly) downgrading.*

Proof: It follows from Equation (8) that

$$g \circ f(x \wedge y) - g \circ f(y) \geq g \circ f(x) - g \circ f(x \vee y) \tag{9}$$

Now, if $f(x) \geq f(x \vee y)$, then $g \circ f(x) \geq g \circ f(x \vee y)$. Therefore, by (9), $g \circ f(x \wedge y) \geq g \circ f(y)$. So, $f(x \wedge y) \geq f(y)$. The proof that $f(x) > f(x \vee y)$ implies $f(x \wedge y) > f(y)$ is analogous. The case of strict supermodularizability is a corollary of Theorem 5. □

This is the strongest possible result, as the following example due to Shannon [8] shows. Let $D = \{0, 1, 2, 3\}$ and $C = \{1, 2, 3, 4, 5\}$. Define $f : D \times \{0, 1\} \to C$ by $f(0, 0) = 1$; $f(1, 0) = 2$; $f(2, 0) = 2$; $f(3, 0) = 1$; $f(0, 1) = 3$; $f(1, 1) = 4$; $f(1, 2) = 5$; $f(1, 3) = 3$. Then f is strictly downgrading but it is not supermodularizable because otherwise there would be an increasing function $g : Co(f) \to \mathbf{R}$ such that $g(4) + g(1) \geq g(3) + g(2) \geq g(5) + g(1)$. This implies $g(4) \geq g(5)$, which contradicts the assumption that g is increasing.

The most important consequence of Theorems 6, 7 and 8 is that they provide very weak sufficient conditions under which (strictly) superassociative and supermedial functions are (strictly) supermodularizable.

Theorem 10 *Let L be a lattice and $f : L \to \mathbf{R}$ a (strictly) superassociative function. If $Co(f)$ is not closed and $*$ is cancellative or if $Co(f)$ is compact and $*$ is a weakly increasing increasing triangular function, then f is (strictly) supermodularizable.*

Proof: From Theorems 6 and 7, it follows that there exists a continuous increasing function $g : Co(f) \to \overline{\mathbf{R}}$ such that $s * t = g^{-1}[g(s) + g(t)]$. By definition of super$*$ function, we have then

$$g^{-1}[g \circ f(x \vee y) + g \circ f(x \wedge y)] \geq g^{-1}[g \circ f(x) + g \circ f(y)].$$

Since the order relation $\geq$ is preserved under increasing transformation, this reduces to $g \circ f(x \vee y) + g \circ f(x \wedge y) \geq g \circ f(x) + g \circ f(y)$. The proof for the case of strict superassociativity is analogous. □

Theorem 11 *Let L be a lattice and $f : L \to \mathbf{R}$ a (strictly) supermedial function. If $*$ is cancellative and idempotent, then f is (strictly) supermodularizable.*

Proof: From Theorem 8, it follows that there exists a continuous increasing function $g : Co(f) \to \mathbf{R}$ such that

$$s * t = g^{-1}\left[\frac{g(s) + g(t)}{2}\right].$$

By definition of super$*$ function, we have then

$$g^{-1}\left[\frac{g \circ f(x \vee y) + g \circ f(x \wedge y)}{2}\right] \geq g^{-1}\left[\frac{g \circ f(x) + g \circ f(y)}{2}\right].$$

Since the order relation $\geq$ is preserved under increasing transformation, this reduces to

$$\frac{g \circ f(x \vee y) + g \circ f(x \wedge y)}{2} \geq \frac{g \circ f(x) + g \circ f(y)}{2}$$

and therefore to $g \circ f(x \vee y) + g \circ f(x \wedge y) \geq g \circ f(x) + g \circ f(y)$. The proof for the case of strict supermedial is analogous. $\square$

This theorem has a quite surprising converse which enlightens the nature of supermodularizable functions.

Theorem 12 *Let L be a lattice and $f : L \to \mathbf{R}$ a (strictly) supermodularizable function. Then f is (strictly) supermedial.*

Proof: Let $m, M \in \overline{\mathbf{R}}$ be such that $Co(f) = <m, M>$. By continuity and increasingness of g, $Ran[g(Co(f))] = <g(m), g(M)>$ and the same brackets of $<m, M>$ apply. Therefore, since $g : Co(f) \to <g(m), g(M)>$, it follows that $g^{-1} : <g(m), g(M)> \to Co(f)$ exists; moreover, it is continuous and increasing. In particular, if s, t are in $<g(m), g(M)>$, then $(s+t)/2$ is also in $<g(m), g(M)>$. Choose arbitrary $x, y \in L$; by supermodularizability, it follows that $g \circ f(x \vee y) + g \circ f(x \wedge y) \geq g \circ f(x) + g \circ f(y)$. Dividing by 2 and applying the map g^{-1} to the resulting inequality, we have

$$g^{-1}\left[\frac{g \circ f(x \vee y) + g \circ f(x \wedge y)}{2}\right] \geq g^{-1}\left[\frac{g \circ f(x) + g \circ f(y)}{2}\right] \tag{10}$$

Now, define $*$ on $Co(f)$ by $s * t = g^{-1}\left[(g(s) + g(t))/2\right]$. Then $*$ is medial, continuous and weakly increasing; hence, Equation (10) establishes supermediality. The proof for the case of strict supermodularizability is analogous. $\square$

A similar but weaker result can be given for superassociative functions.

Theorem 13 *Let L be a lattice and $f : L \to \mathbf{R}$ a function (strictly) supermodularizable by a continuous increasing function $g : Co(f) \to \mathbf{R}$ such that $Ran[g(Co(f))]$ is closed under addition. Then f is (strictly) superassociative.*

Proof: The proof proceeds exactly as the one for Theorem 12.The only difference is that the additional assumption is needed to establish that if $s, t \in <g(m), g(M)>$, then $s + t \in <g(m), g(M)>$ as well. Given this, choose arbitrary $x, y \in L$. It follows by supermodularizability that $g \circ f(x \vee y) + g \circ f(x \wedge y) \geq g \circ f(x) + g \circ f(y)$. Defining $*$ on $Co(f)$ by $s * t = g^{-1}\left[g(s) + g(t)\right]$ and applying the map g^{-1} to this inequality, superassociativity follows. The proof for the case of strict supermodularizability, as usual, is analogous. $\square$

3.4 Superextremal functions

In this section, we consider two classes of functions which, as discussed in Section 3.1, are not included in our definition of the class of super$*$ functions.

Definition 8 *Let L be a lattice. We say that the function $f : L \to Ran(f)$ is supermaximal if*

$$f(x \vee y) \vee f(x \wedge y) \geq f(x) \vee f(y) \tag{11}$$

for all $x, y \in L$ and that it is superminimal if

$$f(x \vee y) \wedge f(x \wedge y) \geq f(x) \wedge f(y) \tag{12}$$

for all $x, y \in L$. If the inequalities hold strictly for all $x \| y \in L$, we say respectively that f is strictly supermaximal or strictly superminimal.

If a function is either supermaximal or superminimal, we will call it a *superextremal function*. A strictly superextremal function is analogously defined. An alternative important characterization of superminimal functions is that all their upper level sets are sublattices.

It is easy to see that superextremal functions fail to belong to the class of symmetric super$*$ functions only because their sections are nondecreasing without being weakly increasing. In fact, this is also the only reason for which they are not included in the smaller classes of superassociative and supermedial functions, although they can certainly be thought of as limiting cases of these. Thus, it is not surprising that superextremal functions bear strong relationships with the downgrading property.

Our first result is the analogue of Theorem 5 for superextremal functions.

Theorem 14 *Let L be a lattice. If $f : L \to \mathbf{R}$ is a superextremal function, then f is quasidowngrading. Moreover, if f is strictly superextremal, then it is strictly downgrading.*

Proof: We prove the counterpositive. Assume f is not quasidowngrading. Then, there exist x, y such that $f(x) > f(x \vee y)$ and $f(y) > f(x \wedge y)$. For $* = \vee$ or $* = \wedge$, it follows that $f(x \vee y) * f(x \wedge y) < f(x) * f(y)$ and f is not superextremal. The proof for the strict case is analogous. □

As for Theorem 5, this is again the strongest possible result. In fact, the example following Theorem 5 suffices to show that a superminimal function is neither meet nor join-downgrading. And if we change its y-values from 0's to 3's, the same example applies to the case of a supermaximal function. Thus, we conclude that a superextremal function may not necessarily be downgrading.

Theorem 14 has a partial converse, which shows that the superextremal property is a local necessary condition for a function to be quasidowngrading. Notice that this implies in particular that a super$*$ or *a fortiori* a supermodularizable function is always locally superextremal.

Theorem 15 *Let L be a lattice. If $f : L \to \mathbf{R}$ is quasidowngrading, then it is superextremal at x, y for any $x, y \in L$. If f is strictly downgrading, then it is strictly superextremal at x, y for any $x, y \in L$.*

Proof: Suppose that f is quasidowngrading and it is not supermaximal at x, y. We show that it has to be superminimal at x, y. If f is not supermaximal at x, y, we have $f(x) \vee f(y) > f(x \vee y) \vee f(x \wedge y)$. Then $f(x) \neq f(y)$, otherwise f would not be quasidowngrading. So, assume without loss of generality that $f(x) > f(y)$. By the quasidowngrading property, $f(x \wedge y) \geq f(y)$. To conclude that f is superminimal at x, y, we only need to show that $f(x \vee y) \geq f(y)$. Suppose this is not so; then the quasidowngrading property would imply $f(x \wedge y) \geq f(x)$, which is impossible. Hence, we conclude that f is superminimal.

The proof for the case of a strictly downgrading function is analogous. □

As the following example shows, this is the strongest possible result. Let $f : \{0,1\}^2 \to \{0,1\}$ be defined by $f(0,0) = 1$; $f(1,0) = 0$; $f(0,1) = 1$; $f(1,1) = 0$. Then f is downgrading but it is not strictly superextremal.

3.5 Differentiable supermodularizable functions

Although they certainly do not exhaust the class of downgrading functions, supermodularizable functions constitute a prominent subclass of these. It is therefore interesting to investigate if there exists a simple sufficient criterion to recognize them.

Throughout this section, we will make the assumption that the lattice L is the direct product of a finite number n of chains and that the function $f : L \to \mathbf{R}$ is twice differentiable everywhere on L. Moreover, given a matrix $A = [a_{ij}]$, we say that A is a *Metzler matrix* if $a_{ij} \geq 0$, for all $i \neq j$. If all the inequalities hold strictly, we say that A is strictly Metzler.

The first part of the following result is proved in Topkis [9]. The second part of it can be easily obtained following his proof.

Theorem 16 *Let L be an interval in $\mathbf{R}^n$ and let the function $f : L \to \mathbf{R}$ be twice differentiable on an open superset of L. Then f is supermodular if and only if its Hessian matrix $\nabla^2 f(x)$ is Metzler for any $x \in L$. If $\nabla^2 f(x)$ is strictly Metzler for any $x \in L$, then f is strictly supermodular.*

From this theorem, we immediately obtain the following sufficient condition for the supermodularizability of a function f on L.

Theorem 17 *Let L be an interval in $\mathbf{R}^n$ and let the function $f : L \to \mathbf{R}$ be twice differentiable on an open superset of L. If there exists a twice differentiable and increasing function $g : Co(f) \to \mathbf{R}$ such that the matrix*

$$G(x) = g'[f(x)]\nabla^2 f(x) + g''[f(x)]\nabla f(x)\nabla f(x)^\top$$

is (strictly) Metzler for all $x \in L$, then f is (strictly) supermodularizable.

We now introduce a subclass of supermodularizable functions to which this result can be fruitfully applied to obtain a simple characterization.

Definition 9 *Let L be a lattice. For $k \in \mathbf{R}$, we say that the function $f : L \to \mathbf{R}$ is k-supermodular if it is supermodular ($k = 0$) or if there exists some $k \neq 0$ such that*

$$\operatorname{sgn}(k)\left\{\exp[kf(x \vee y)] + \exp[kf(x \wedge y)]\right\} \geq \operatorname{sgn}(k)\left\{\exp[kf(x)] + \exp[kf(y)]\right\} \qquad (13)$$

for all $x, y \in L$. If the inequality holds strictly for all $x \| y \in L$, we say that f is strictly k-supermodular.

It is easy to see that a (strictly) k-supermodular function is supermodularizable if and only if it is supermodularizable by $g(s) = \operatorname{sgn}(k)\exp(ks)$. By Theorem 12, a (strictly) k-supermodular functions is (strictly) supermedial for

$$s *_k t = \log\left[\frac{\exp(ks) + \exp(kt)}{2}\right]^{1/k}$$

which is also known as the exponential mean of order k. Since $\lim_{k\downarrow-\infty} s *_k t = s \wedge t$, $\lim_{k\to 0} s *_k t = (s+t)/2$ and $\lim_{k\uparrow\infty} s *_k t = s \vee t$, as k increases the k-supermodular functions range from superminimal ($k = -\infty$) to supermaximal ($k = \infty$) passing through the supermodular functions ($k = 0$).

The main result of this section is the following characterization for k-supermodular functions. Its proof is an immediate consequence of Theorem 16 and 17 and of the observation that a k-supermodular function is supermodularizable by $g(s) = \operatorname{sgn}(k)\exp(ks)$.

Theorem 18 *Let L be an interval in $\mathbf{R}^n$ and let the function $f : L \to \mathbf{R}$ be twice differentiable on an open superset of L. Then f is k-supermodular if and only if there exists $k \in \mathbf{R}$ such that the matrix*

$$K(x) = \nabla^2 f(x) + k\,\nabla f(x)\nabla f(x)^\top \qquad (14)$$

is Metzler for any $x \in L$. If $K(x)$ is strictly Metzler for any $x \in L$, then f is strictly k-supermodular.

Some immediate results can be drawn from Theorem 18. Let H denote the matrix given by $H_{ij} = 1$ if $i \neq j$ and 0 otherwise.

Theorem 19 *Let L be an interval in $\mathbf{R}^n$ and let the function $f : L \to \mathbf{R}$ be twice differentiable on an open superset of L. Suppose there exist $m \in \mathbf{R}$ and $\epsilon > 0$ such that for all $x \in L$*

(*Hessian Metzler bounded from below*)	$\nabla^2 f(x) + mH$ *is Metzler*
($\nabla f \nabla f^\top$ *Metzler positively bounded away from zero*)	$\nabla f(x)\nabla f(x)^\top - \epsilon H$ *is Metzler*

Then f is k-supermodular for $k = \max\{0, m/\epsilon\}$ and strictly k-supermodular for $k > \max\{0, m/\epsilon\}$.

Proof: By Theorem 18, it suffices to show that $K(x)$ is (strictly) Metzler for any $x \in L$. The boundedness assumptions guarantee that $\nabla^2 f(x) + k\,\nabla f(x)\nabla f(x)^\top + (m - k\epsilon)H$ is (strictly) Metzler for the given choices of k, which establishes the result. $\square$

Notice that if $m \leq 0$ then f is supermodular. If $\nabla f(x)\nabla f(x)^\top$ is Metzler negatively bounded away from zero by $\epsilon < 0$, Theorem 19 holds for $k \leq \min\{0, m/\epsilon\}$.

Corollary 20 *Let L be an interval in $\mathbf{R}^n$ and let the function $f : L \to \mathbf{R}$ be twice differentiable on an open superset of L. If L is compact in the standard topology on $\mathbf{R}^n$ and*

$$\frac{\partial f}{\partial x_i} > 0 \qquad \text{for all } i = 1, \ldots, n \tag{15}$$

then f is supermodularizable.

Proof: By compactness, the Hessian matrix is bounded below and the gradient is bounded away from 0 in the Metzler sense. Therefore, there exists some k such that f is (strictly) k-supermodular and thus (strictly) supermodularizable. $\square$

Notice that we could replace (15) by the assumption that the partial derivatives are all negative.

Theorem 19 provides a sufficient condition to recognize a (strictly) supermodularizable function, and therefore guarantees that such function has the (strict) downgrading property. We now provide an example in which it applies to establish that even a strictly submodular function may satisfy the downgrading property. Let $D = (-\infty, -1] \times [1/2, 1]$. Define $f : D \to \mathbf{R}^-$ to be $f(x_1, x_2) = x_1/x_2$. Then $f_{12}(x) = -x_2^{-2} < 0$ for all $x \in D$ and therefore f is strictly submodular. However, since $f_1(x) = x_2^{-1} \geq 1$, $f_2(x) = -x_1 x_2^{-2} \geq 1$, and $f_{12}(x) = -x_2^{-2} \geq -4$, we have that $m = 4$ and $\epsilon = 1$ apply in Theorem 19 to obtain that f is (strictly) k-supermodular for $k = 4$ ($k > 4$).

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